Checking Termination of Logic Programs with Function Symbols Through Linear Constraints

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RuleML 2014 - Prague

August 18-20, 2014

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The bottom-up evaluation of this program never terminates, producing $nat(0), nat(s(0)), nat(s(s(0))), \ldots$.

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- Establishing if a program has a terminating bottom-up evaluation is an **undecidable problem**;
- Recent work has focused on finding sufficient conditions for the termination of logic programs.

Argument-restricted criterion [Lierler&Lifschitz, ICLP'09]

Computes, for each predicate argument, an upper bound of the depth of terms that may occur in that argument.

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q[1] → 0, p[1] → 1

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- **Pros**:
  - Simple: it just needs to compute a particular level mapping for each argument;
  - Efficient: Computing an argument ranking (if exists) requires polynomial time.
- **Cons**:
  - Only characterizes **the depth of terms** inside arguments **alone**;
  - No distinction between different function symbols;
  - Very **few practical (terminating) programs** are recognized.

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- Very simple notion of atom size (we can do better): sum all the terms depths.
    - $\mathtt{size}(\mathtt{head}(\mathtt{r}_2)) = 0 + 1$;
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- The overall size of the head will never increase **w.r.t. the body**.

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If *all* rules satisfy this condition, the program is terminating.

Firing Graph

Not all rules that increase the head size are **dangerous**.

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r_1: & \mathtt{p(f(X))} & \leftarrow & \mathtt{q(X)}. \\
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However, rule r_1 will never "trigger" r_2.

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Definition (Firing Graph)

Directed graph whose nodes are the rules of the program and there is an edge from r_1 to r_2 if r_1 fires r_2.

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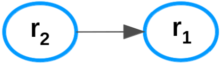

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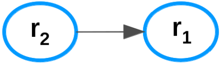

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There is no cyclic execution of r_1 and r_2.

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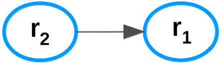
$$r_1 : \mathtt{p(}\overset{2}{\mathtt{f(X)}}\mathtt{)} \leftarrow \mathtt{q(}\overset{1}{\mathtt{X}}\mathtt{)}.$$

$$r_2 : \mathtt{q(}\overset{1}{\mathtt{f(X)}}\mathtt{)} \leftarrow \mathtt{p(}\overset{1}{\mathtt{g(X)}}\mathtt{)}.$$



The program is not argument-restricted.

However, rule r_1 will never "trigger" r_2.

Definition (Firing Graph)

Directed graph whose nodes are the rules of the program and there is an edge from r_1 to r_2 if r_1 fires r_2.

There is no cyclic execution of r_1 and r_2.

Goal: Check termination of the SCCs of the Firing Graph.

Term size

Intuition: The size of a term is a template for all possible sizes the term may have during the program evaluation.

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size(t,r) is the size of term t w.r.t. rule r

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The size of an atom is a **linear combination of its terms sizes**.

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Given a rule r and an atom $A = p(t_1, \ldots, t_n)$ in r, the **size of** A is:

$$size(A, r) = \alpha_{p_1} \cdot size(t_1, r) + \ldots + \alpha_{p_n} \cdot size(t_n, r)$$

where $\alpha_{p_1}, \ldots, \alpha_{p_n}$ are **positive integers**.

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We allow for the coefficients α_i to be not fixed a priori.

They will be chosen depending on the program structure.

Rule-bounded programs

Definition

A program $\mathcal{P}$ is *rule-bounded* if for every SCC $\mathcal{C}$, every rule r in $\mathcal{C}$ is such that:

$$size(body(r), r) \geq size(head(r), r)$$

for some fixed set of parameters α_i.

Rule-bounded programs

Example

Consider the length program:

$$\begin{array}{ll} \mathtt{r}_1: & \mathtt{len}([\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}],\mathtt{0}). \\ \mathtt{r}_2: & \mathtt{len}(\mathtt{Tail},\mathbf{s}(\mathtt{N})) \leftarrow \mathtt{len}(\mathbf{list}(\mathtt{Head},\mathtt{Tail}),\mathtt{N}). \end{array}$$

Rule-bounded programs

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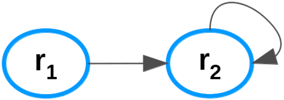

Figure: Firing Graph

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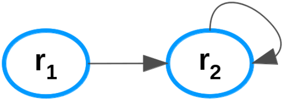

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The only SCC is $\mathcal{C} = \{r_2\}$, thus we need to check:

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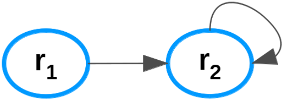

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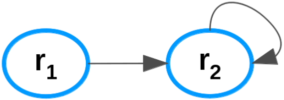

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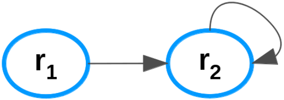

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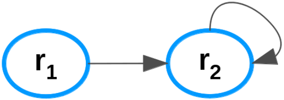

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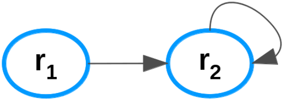

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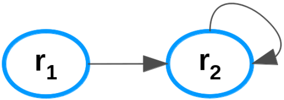

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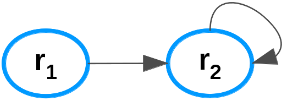

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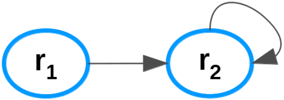

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We can choose $\alpha_1 = \alpha_2 = 1 \Rightarrow$ the program is rule-bounded.

Rule-bounded programs

Theorem

The bottom-up evaluation of a rule-bounded program always terminates.

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Theorem

The complexity of checking whether a program is rule-bounded is NP.

Examples of rule-bounded programs

Example (Bubble sort)

$$\begin{array}{llcl}
r_0: & \texttt{sort}(\texttt{List}, [\,], [\,]) & \leftarrow & \texttt{input}(\texttt{List}). \\
r_1: & \texttt{sort}([\texttt{Y}|\texttt{T}], [\texttt{X}|\texttt{Temp}], \texttt{Sorted}) & \leftarrow & \texttt{sort}([\texttt{X}|[\texttt{Y}|\texttt{T}]]), \texttt{Temp}, \texttt{Sorted}), \texttt{X} \leq \texttt{Y}. \\
r_2: & \texttt{sort}([\texttt{X}|\texttt{T}], [\texttt{Y}|\texttt{Temp}], \texttt{Sorted}) & \leftarrow & \texttt{sort}([\texttt{X}|[\texttt{Y}|\texttt{T}]]), \texttt{Temp}, \texttt{Sorted}), \texttt{Y} < \texttt{X}. \\
r_3: & \texttt{sort}(\texttt{Temp}, [\,], [\texttt{X}|\texttt{Sorted}]) & \leftarrow & \texttt{sort}([\texttt{X}]), \texttt{Temp}, \texttt{Sorted}).
\end{array}$$

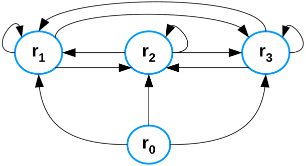

Figure: Bubble sort Firing Graph

$$\text{SCC } \mathcal{C} = \{r_1, r_2, r_3\}$$

Examples of rule-bounded programs

Example (Bubble sort)

$$\begin{array}{llcl}
r_0: & \mathtt{sort(List,[\,],[\,])} & \leftarrow & \mathtt{input(List)}. \\
r_1: & \mathtt{sort([Y|T],[X|Temp],Sorted)} & \leftarrow & \mathtt{sort([X|[Y|T]]),Temp,Sorted),X \le Y}. \\
r_2: & \mathtt{sort([X|T],[Y|Temp],Sorted)} & \leftarrow & \mathtt{sort([X|[Y|T]]),Temp,Sorted),Y < X}. \\
r_3: & \mathtt{sort(Temp,[\,],[X|Sorted])} & \leftarrow & \mathtt{sort([X]),Temp,Sorted)}.
\end{array}$$

$$\begin{cases}
\alpha_1 \cdot (4+x+y+t) + \alpha_2 \cdot temp + \alpha_3 \cdot sorted \ge \\
\qquad \alpha_1 \cdot (2+y+t) + \alpha_2 \cdot (2+x+temp) + \alpha_3 \cdot sorted \\
\alpha_1 \cdot (4+x+y+t) + \alpha_2 \cdot temp + \alpha_3 \cdot sorted \ge \\
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r_3: & \texttt{sort(Temp, [], [X|Sorted])} & \leftarrow & \texttt{sort([X]), Temp, Sorted)}.
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\qquad \alpha_1 \cdot temp + \alpha_2 \cdot 0 + \alpha_3 \cdot (2+x+sorted)
\end{cases}
$$

A possible solution is $\alpha_1 = 2$, $\alpha_2 = 2$, $\alpha_3 = 1$

Examples of rule-bounded programs

Example (Tree visit)

r_0 : $\mathtt{visit(Tree,[\,],[\,])} \leftarrow \mathtt{input(Tree)}.$
r_1 : $\mathtt{visit(Left,[Root|Visited],[Right|ToVisit])} \leftarrow$
$\mathtt{visit(tree(Root,Left,Right),Visited,ToVisit)}.$
r_2 : $\mathtt{visit(Next,Visited,ToVisit)} \leftarrow \mathtt{visit(null,Visited,[Next|ToVisit])}.$

$$\begin{cases} \alpha_1 \cdot (3 + root + left + right) + \alpha_2 \cdot visited + \alpha_3 \cdot tovisit \geq \\ \qquad \alpha_1 \cdot left + \alpha_2 \cdot (2 + root + visited) + \alpha_3 \cdot (2 + right + tovisit) \\ \alpha_1 \cdot 0 + \alpha_2 \cdot visited + \alpha_3 \cdot (2 + next + tovisit) \geq \\ \qquad \alpha_1 \cdot next + \alpha_2 \cdot visited + \alpha_3 \cdot tovisit \end{cases}$$

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\end{aligned}
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$$
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\end{cases}
$$

A possible solution is $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_3 = 2$

Dealing with non-increasing cycles

Example

$$\begin{array}{lll} r_1: \mathtt{p(X,Y)} & \leftarrow & \mathtt{q(f(X),Y)}. \\ r_2: \mathtt{q(W,f(Z))} & \leftarrow & \mathtt{p(W,Z)}. \end{array}$$

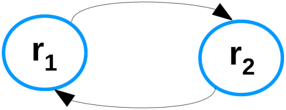

Figure: Firing Graph

The program terminates, but...

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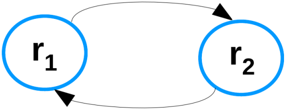

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The program terminates, but...

Rule r_2 increases its head size $\Rightarrow$ program is not rule-bounded.

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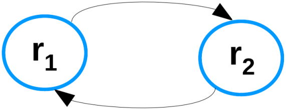

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The program terminates, but...

Rule r_2 increases its head size ⇒ program is not rule-bounded.

But the **cycle** does not increase the size of propagated values.

Dealing with non-increasing cycles

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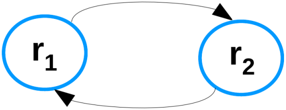

Figure: Firing Graph

We can rewrite the cycle r_1, r_2 as a single rule:

Dealing with non-increasing cycles

Example

$$\begin{aligned} r_1 :\ & \mathtt{p(X,Y)} && \leftarrow && \mathtt{q(f(X),Y).} \\ r_2 :\ & \mathtt{q(W,f(Z))} && \leftarrow && \mathtt{p(W,Z).} \end{aligned}$$

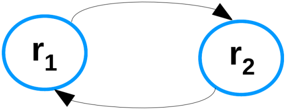

Figure: Firing Graph

We can rewrite the cycle r_1, r_2 as a single rule:

$$\underbrace{\mathtt{q(f(X),Y)} \rightarrow \mathtt{p(X,Y)}}_{r_1}$$

Dealing with non-increasing cycles

Example

$$\begin{array}{llcl} r_1: & \mathtt{p(X,Y)} & \leftarrow & \mathtt{q(f(X),Y)}. \\ r_2: & \mathtt{q(W,f(Z))} & \leftarrow & \mathtt{p(W,Z)}. \end{array}$$

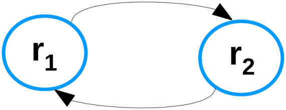

Figure: Firing Graph

We can rewrite the cycle r_1, r_2 as a single rule:

$$\underbrace{\mathtt{q(f(X),Y)} \rightarrow \mathtt{p(X,Y)}}_{r_1} \Rightarrow \underbrace{\mathtt{p(W,Z)} \rightarrow \mathtt{q(W,f(Z))}}_{r_2}$$

Dealing with non-increasing cycles

Example

$$
\begin{array}{llcl}
r_1: & \mathtt{p(X,Y)} & \leftarrow & \mathtt{q(f(X),Y).} \\
r_2: & \mathtt{q(W,f(Z))} & \leftarrow & \mathtt{p(W,Z).}
\end{array}
$$

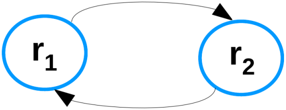

Figure: Firing Graph

We can rewrite the cycle r_1, r_2 as a single rule:

$$\underbrace{\mathtt{q(f(X),Y)} \rightarrow \color{red}{\mathtt{p(X,Y)}}}_{r_1} \Rightarrow \underbrace{\color{red}{\mathtt{p(W,Z)}} \rightarrow \mathtt{q(W,f(Z))}}_{r_2}$$

Dealing with non-increasing cycles

Example

$$
\begin{aligned}
r_1 : \ & \mathtt{p(X,Y)} & \leftarrow \ & \mathtt{q(f(X),Y).} \\
r_2 : \ & \mathtt{q(W,f(Z))} & \leftarrow \ & \mathtt{p(W,Z).}
\end{aligned}
$$

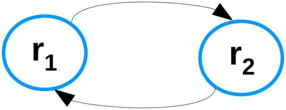

Figure: Firing Graph

We can rewrite the cycle r_1, r_2 as a single rule:

$$\underbrace{\mathtt{q(f(X),Y)} \rightarrow \mathtt{p(X,Y)}}_{r_1} \Rightarrow \underbrace{\mathtt{p(W,Z)} \rightarrow \mathtt{q(W,f(Z))}}_{r_2}$$

$$\mathtt{X/W,\ Y/Z}$$

Dealing with non-increasing cycles

Example

$$\begin{array}{lll} r_1: \mathtt{p(X,Y)} & \leftarrow & \mathtt{q(f(X),Y)}. \\ r_2: \mathtt{q(W,f(Z))} & \leftarrow & \mathtt{p(W,Z)}. \end{array}$$

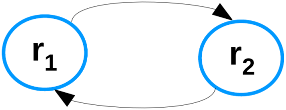

Figure: Firing Graph

We can rewrite the cycle r_1, r_2 as a single rule:

$$\underbrace{\mathtt{q(f(X),Y)} \rightarrow \mathtt{p(X,Y)}}_{r_1} \Rightarrow \underbrace{\mathtt{p(W,Z)} \rightarrow \mathtt{q(W,f(Z))}}_{r_2}$$

$$\mathtt{X/W,\ Y/Z}$$

$$\underbrace{\mathtt{q(f(W),Z)} \rightarrow \mathtt{q(W,f(Z))}}_{r_{12}}$$

Dealing with non-increasing cycles

Example

$$\begin{array}{lll} r_1: \ \mathtt{p(X,Y)} & \leftarrow & \mathtt{q(f(X),Y).} \\ r_2: \ \mathtt{q(W,f(Z))} & \leftarrow & \mathtt{p(W,Z).} \end{array}$$

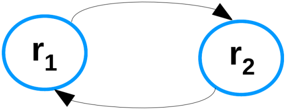

Figure: Firing Graph

We can rewrite the cycle r_1, r_2 as a single rule:

$$\underbrace{\mathtt{q(f(X),Y)} \rightarrow \mathtt{p(X,Y)}}_{r_1} \Rightarrow \underbrace{\mathtt{p(W,Z)} \rightarrow \mathtt{q(W,f(Z))}}_{r_2}$$

$$\mathtt{X/W,\ Y/Z}$$

$$\underbrace{\mathtt{q(f(W),Z)} \rightarrow \mathtt{q(W,f(Z))}}_{r_{12}}$$

$$\alpha_{q_1} \cdot (1+w) + \alpha_{q_2} \cdot z \geq \alpha_{q_1} \cdot w + \alpha_{q_2} \cdot (1+z)$$

Dealing with non-increasing cycles

Example

$$
\begin{array}{llll}
r_1: & \mathtt{p(X, Y)} & \leftarrow & \mathtt{q(f(X), Y).} \\
r_2: & \mathtt{q(W, f(Z))} & \leftarrow & \mathtt{p(W, Z).}
\end{array}
$$

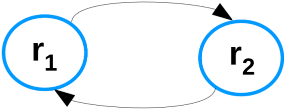

Figure: Firing Graph

We can rewrite the cycle r_1, r_2 as a single rule:

$$\underbrace{\mathtt{q(f(X), Y)} \rightarrow \mathtt{p(X, Y)}}_{r_1} \Rightarrow \underbrace{\mathtt{p(W, Z)} \rightarrow \mathtt{q(W, f(Z))}}_{r_2}$$

$$\mathtt{X/W,\ Y/Z}$$

$$\underbrace{\mathtt{q(f(W), Z)} \rightarrow \mathtt{q(W, f(Z))}}_{r_{12}}$$

$$\alpha_{q_1} \cdot (1 + w) + \alpha_{q_2} \cdot z \geq \alpha_{q_1} \cdot w + \alpha_{q_2} \cdot (1 + z)$$

$$\alpha_{q_1} \geq \alpha_{q_2}$$

Dealing with non-increasing cycles

Example

$$\begin{array}{llcl} r_1: & \mathtt{p(X,Y)} & \leftarrow & \mathtt{q(f(X),Y).} \\ r_2: & \mathtt{q(W,f(Z))} & \leftarrow & \mathtt{p(W,Z).} \end{array}$$

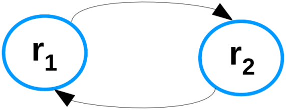

Figure: Firing Graph

We can rewrite the cycle r_1, r_2 as a single rule:

$$\underbrace{\mathtt{q(f(X),Y)} \rightarrow \mathtt{p(X,Y)}}_{r_1} \Rightarrow \underbrace{\mathtt{p(W,Z)} \rightarrow \mathtt{q(W,f(Z))}}_{r_2}$$

$$\mathtt{X/W,\ Y/Z}$$

$$\underbrace{\mathtt{q(f(W),Z)} \rightarrow \mathtt{q(W,f(Z))}}_{r_{12}}$$

$$\alpha_{q_1} \cdot (1+w) + \alpha_{q_2} \cdot z \geq \alpha_{q_1} \cdot w + \alpha_{q_2} \cdot (1+z)$$

$$\alpha_{q_1} \geq \alpha_{q_2} \qquad\qquad \alpha_{q_1} = \alpha_{q_2} = 1$$

Conclusion

Contributions:

- Using linear constraints for checking bottom-up termination;
- The technique is complementary to the other techniques that analyze single arguments;
- The technique recognizes a good number of practical logic programs;

Future work:

- Combine rule-bounded programs with other techniques in the literature;
- Deep complexity analysis of the proposed technique (there may be many tractable cases);
- Study the termination problem for programs with interpreted function symbols (none of the currently known techniques support them).

Some reference

1. Bounded programs: a new decidable class of logic programs with function symbols, *Greco et al.*, IJCAI '13.
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5. One more decidable class of finitely ground programs, *Yuliya Lierler and Vladimir Lifschitz*, ICLP '09.
6. Computable functions in ASP: Theory and Implementation, *Calimeri et al.*, ICLP '08.
7. Termination detection in logic programs using argument sizes, *Kirack Sohn and Allen Van Gelder*, PODS '91.

Thank you for your attention!